Two hard problems in box-Totally Dual Integral polyhedra

Patrick Chervet[1], Roland Grappe[2], Mathieu Lacroix[2],
Francesco Pisanu[2], Roberto Wolfler-Calvo[2]

[1] Lycée Olympe de Gouges, rue de Montreuil à Claye, 93130, Noisy le Sec, France.
[2] Université Sorbonne Paris Nord, LIPN, CNRS UMR 7030, F-93430, Villetaneuse, France.
{grappe,lacroix,pisanu,wolfler}@lipn.fr

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1 Introduction

A rational linear system is *totally dual integral (TDI)* if for every integer linear function for which the optimum is finite the associated dual problem has an integer optimal solution. A TDI system is *box-TDI* if adding any rational bounds on the variables preserves its TDIness. Box-TDI systems are systems that yield strong min-max relations such as the one involved in the Max Flow-Min Cut Theorem of Ford and Fulkerson.

Box-total dual integral systems and polyhedra received a lot of attention from the combinatorial optimization community around the 80's. A renewed interest appeared in the last decade and since then many deep results appeared involving such systems.

Box-TDI systems characterize totally unimodular matrices. A matrix is *totally unimodular (TU)* if every subset of linearly independent rows forms a unimodular matrix, a matrix being *unimodular* if it has full row rank and all its nonzero maximal minors have value ± 1. A matrix A is TU if and only if the system $Ax \leq b$ is box-TDI for each rational vector b [7, Page 318].

Until recently, the vast majority of known box-TDI systems were systems associated with TU matrices. For instance, König's Theorem [6] can be seen as a consequence of the fact that the vertex-edge incidence matrix of a graph is TU if and only if the graph is bipartite [5].

Box-TDI systems do not describe all rational polyhedra, while TDI systems do [4]. A polyhedron that can be described by a box-TDI system is a *box-TDI polyhedron*, and every TDI system describing it is actually box-TDI [2]. Box-TDI polyhedra characterize the following generalization of TU matrices. A matrix is *totally equimodular (TE)* if every subset of linearly independent rows forms an equimodular matrix, a matrix being *equimodular* if it has full row rank and all its nonzero maximal minors have the same absolute value. A matrix A is TE if and only if the polyhedron $\{x \colon Ax \leq b\}$ is box-TDI for each rational vector b [1].

2 Contributions

In this work, we prove that the problem of deciding whether a given polyhedron is box-TDI is co-NP-complete. Our proof builds upon the hardness result of Ding et al. [3] about the recognition of box-TDI systems.

We also prove that the edge-vertex incidence matrix of any graph is TE. This implies that the edge relaxation of the stable set problem is a box-TDI polyhedron. From the NP-hardness of the maximum stable set problem, it follows that optimizing a linear function over $\{x \in \mathbb{Z}^n \colon Ax \leq \mathbf{1}\}$ is NP-hard when A is TE. Since the latter problem is polynomial when A is TU, this unveils a major difference between TE and TU matrices. Moreover, this hardness result also implies that integer optimization over box-TDI polyhedra is NP-hard.

Références

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