Locally stable exchanges - ROADEF 2023

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1 Introduction

We extend recent work on stable kidney exchanges by introducing and underlining the relevance of a new concept of locally stable exchanges. We link this new concept to local kernels in digraphs.

2 Abstract

Kidney exchange programmes offer an attractive option for the treatment of patients with an end-stage renal disease, by allowing them to receive a kidney transplant from a living donor. The maximization of the number of kidney exchanges within a pool of potential donors and recipients is a combinatorial optimization problem which has generated an abundant literature over the past 20 years, in the footprints of a seminal paper by Roth et al. [4]. Several authors have also pointed out the relevance of the classical literature on stable matchings [2] in this context. The optimization of **stable** kidney **exchanges**, however, has not been widely investigated. We extend recent work by Klimentova et al. [3] on this topic by introducing and underlining the relevance of a new concept of *locally stable exchanges* (L-stable exchanges).

A *compatibility digraph* is a digraph $G = (V, A)$ (representing, for example, feasible kidney transplants) such that for each vertex $i \in V$, a preference order is given on the set of in-neighbors $N^-(i) = \{j : (j, i) \in A\}$. An *exchange* is a cycle packing in G. We introduce several new definitions :

— Given a cycle c in G, a *potential L-blocking cycle* for c is a cycle c' which intersects c and such that each vertex of c' either is not in c, or prefers its predecessor in c' to its predecessor in c.

— Given an exchange $\mathcal{M}$ in G, an *L-blocking cycle* for $\mathcal{M}$ is a cycle c that is not included in $\mathcal{M}$ but that has a vertex in common with $\mathcal{M}$, and such that, for every vertex i in cycle c, either i is not in any cycle of $\mathcal{M}$, or i prefers its predecessor in c to its predecessor in c', where $i \in c' \in \mathcal{M}$.

— An *L-stable exchange* is an exchange without L-blocking cycles.

Then, with the compatibility digraph $G = (V, A)$, we associate a *blocking digraph* $G^* = (V^*, A^*)$ such that :

— The vertex set V^* is the set of cycles of G.

— An arc (u, v) is in A^* if the cycle u intersects the cycle v in G and u does not potentially block v.

We show that locally stable exchanges in G correspond exactly to so-called *local kernels* (L-kernels) in G^* (whereas stable exchanges in G correspond to *kernel* in G^*). The notion of local kernel has been considered earlier in graph theory (see, e.g., [1]). However, the literature on local kernels is scarce and does not seem to mention any algorithmic or numerical contributions.

We prove that it is NP-hard to determine whether a graph has a nonempty local kernel, and, hence, to find a local kernel of maximum cardinality. Next, we propose several integer programming formulations for the maximum L-stable exchange problem. These formulations can simply be viewed as modeling the maximum L-kernel problem in a digraph. We also numerically compare these formulations. The results show that the most efficient formulation (with regard to computing time) is one where there is a cycle packing constraint and a stability constraint for each cycle. Other formulations have such constraints for each pair of intersecting cycles. We numerically compare our formulation with a formulation of the maximum stable exchange problem proposed in [3]. Even though the two problems are not equivalent, all the instances tested are such that, when a maximum stable exchange exists, then its cardinality is equal to the cardinality of a maximum L-stable exchange. On the other hand, tests on the instances that do not have a stable exchange reveal that all of these instances have a (nonempty) L-stable exchange. Even more interestingly, for a given instance size, the average cardinality of a maximum L-stable exchange is sometimes very close to the average optimal value over the instances that do have a stable exchange. These observations suggest that the concept of locally stable exchange is indeed interesting, and provides relevant solutions for more instances than the concept of stable exchanges.

All the above results and observations carry over when the concept of *locally stable exchanges* is extended to the concept of *locally strong stable exchanges*.

Références

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