Generalized Nash Fairness solutions for Bi-Objective Minimization Problems

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1 Introduction

In this paper, we consider a special case of Bi-Objective Optimization (BOO), called *Bi-Objective Minimization* (BOM), where two objective functions to be minimized take only positive values. Many applications in telecommunications, logistics, economics, etc. can be formulated as BOM. We can list here several examples like the Bi-Objective Shortest Path Problem where each arc is associated with a cost and a travel time and one desires to compute a shortest path minimizing the total cost and the total travel time between a given source to a given destination, the Bi-Objective Spanning Tree Problem which aims at finding a spanning tree minimizing simultaneously the total cost and the diameter of the tree, etc.

Obviously, popular methods for solving BOO can also be applied for solving BOM. Based on the concept of the Pareto-optimal solutions that are non-dominated with respect to each other, such methods usually construct a representation of the Pareto set that represents different trade-offs between the objectives. They can be mainly divided into two classes, i.e. methods with a posterior articulation of preferences and methods with a priori articulation of preferences. In the former methods such as the Normal Boundary Intersection (NBI), the Non-dominated Sorting Genetic Algorithm-II (NSGA-II), a central decision-maker (CDM) selects manually its own prefered solutions from the Pareto set. Although these methods can provide all the Pareto-optimal solutions, it may be a difficult task due to a huge number of solutions in the Pareto set.

In practice, methods with a priori articulation of preferences, which consider the preferences of the CDM before running the optimization algorithm and then allow the algorithm to determine the solutions that reflect such preferences, have been used more extensively due to their computational efficiency. They usually formulate a single-objective optimization problem whose optimal solutions are the Pareto-optimal solutions to the bi-objective optimization problem. For instance, Weighted Sum Method scalarizes two objectives into a single objective by multiplying each objective with a weight supplied by the CDM [1]. ϵ-constraints Method [2] keeps only one objective and use a CDM-specified value as an upper bound for the other objective. By changing this value, we are able to obtain some different optimal solutions. There are also many other methods such as the lexicographic method, the goal programming, Hypervolume/Chebyshev scalarization, etc. However, most of above approaches for BOM simply outline the methods and show that they can provide efficiently the Pareto-optimal solutions. Essentially, how to determine selection criterion for the preferred solutions and what is the signification of parameters using for the preferences remain the challenging questions.

In this paper, we propose a novel selection criterion for BOM which can guide efficiently the Weighted Sum Method to find the preferred Pareto-optimal solutions achieving some proportional Nash equilibrium between the two objectives in the context of fair competition based on proportional fairness [3], [4], [5]. The latter aims to provide a compromise between the utilitarian rule - which emphasizes overall system efficiency, and the egalitarian rule - which emphasizes individual fairness. In the context of BOM, proportional fairness means that the sum of proportional changes in objectives' value when switching from a preferred solution to any other feasible solutions is not negative. For our purpose, we consider a more general version of proportional fairness in order to take into account the relative importance of one objective to the other according to the point of view of the CDM. More precisely, we introduce the concept of the *generalized Nash Fairness* solution, i.e. ρ-NF solution, for BOM where $\rho > 0$ is a factor denoting the relative importance of the first objective comparing to the second one. This allows the CDM to consider that ρ percent change of the first objective is comparably equivalent to one percent change of the second one. Hence, when switching from a ρ-NF solution to any other feasible solutions, the sum of the factor ρ of the proportional change in the first objective and the proportional change in the second objective is not negative.

Let P, Q represent two objectives of BOM and $\rho > 0$ reflect the relative importance of P to Q. Using this definition of proportional-fair scheduling, a feasible solution $(P^*, Q^*) \in S$ is a ρ-NF solution if and only if

$$\rho \frac{P - P^*}{P^*} + \frac{Q - Q^*}{Q^*} \geq 0 \iff \rho \frac{P}{P^*} + \frac{Q}{Q^*} \geq \rho + 1, \ \forall (P, Q) \in \mathcal{S}, \tag{1}$$

where $\mathcal{S}$ is the set of pairs (P, Q) corresponding to all feasible solutions of BOM.

Note that in the recent conference paper [6], we introduced the notion of NF solution which is a special case of ρ-NF solution when $\rho = 1$ for the Bi-Objective Travelling Salesman Problem (BOTSP). In this paper, we introduce the concept of ρ-NF solution and generalize the theory to BOM.

We first show the existence of ρ-NF solutions for BOM. Furthermore, the set of ρ-NF solutions is a subset of the Pareto set and the inclusion can be strict. As there are possibly many ρ-NF solutions, we focus on extreme ρ-NF solutions having either the smallest value of P or the smallest value of Q. Then, we propose a Newton-like iterative algorithm which converges to extreme ρ-NF solutions in a finite number of iterations. It consists in minimizing a sequence of linear combinations of two objectives based on Weighted Sum Method. Moreover, this algorithm can be modified for finding all ρ-NF solutions. Computational results on several instances of Bi-Objective Shortest Path Problem will be presented and commented.

Références

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