Polyhedral approaches and bounding sets for bi-objective linear binary programming

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1 Introduction to multi-objective combinatorial optimization

Most real-world combinatorial optimization problems actually involve multiple goals. For instance, the shortest path problem is $\mathcal{NP}$-complete even considering only two objectives [1]. What makes the multi-objective integer programming (MOIP) much harder is that the different objective functions generally are conflicting and inconsistent, i.e. there exists no solution optimizing every objective simultaneously. As thus in a multi-objective context, what we look for is the complete set of *Pareto optimal* [2] (so called *efficient*) solutions such that solutions cannot be improved in any objective without losing advantages in at least one of other objectives. Moreover as there might be an exponential number of efficient solutions [3], the MOIP problem is generally $\mathcal{NP}$-hard and intractable. Without loss of generality, consider the following MOIP

$$\begin{aligned} \min_x \quad & z(x) = (z_1(x), \dots, z_p(x)) \\ \text{s.t.} \quad & x \in \mathcal{X} = \{x \in \mathbb{N}^n : Ax \leqslant b\} \end{aligned}$$

where $p \geqslant 2$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Denote $\mathcal{Y} := z(\mathcal{X})$, the image of the set of feasible solutions, $\mathcal{Y}_N := z(\mathcal{X}_E)$ the non-dominated set that is the image of efficient solutions.

In the literature, exact algorithms for multi-objective optimization problems are mainly divided in two categories : the criterion space search scheme and decision space search scheme. Under the criterion space searching, many algorithms explore the objective space and iteratively solve a (mono-objective) IP problem by invoking an IP solver : this is the case of the dichotomy method [4], ϵ-constraint algorithm [5] or the boxed line method [6]. Despite taking advantage of modern commercial solver, criterion space searching algorithms might be highly time consuming, since the IP problems are generally $\mathcal{NP}$-hard.

As to the decision space exploration, the only known framework is the multi-objective Branch&Bound (MOB&B) scheme [7]. The advantage of MOB&B is that a fully implicit enumeration ensures a complete outcome $\mathcal{Y}_N$ and can be easily generalised in many-objective context. It is crucial to strengthen the relaxation bound for accelerating the enumeration searching progress.

Therefore in this work, we propose a first multi-point cutting plane scheme applied in the bi-objective B&B algorithm.

2 Bi-objective Branch&Bound algorithm

Single-objective B&B or BOB&B splits sequentially on variable domain and evaluates subproblems within a tree search process. Unlike mono-objective B&B, the non-dominated set is bounded by two bound sets, the lower bound set (LBS) and the upper bound set (UBS) consisting of a set of points in $\mathbb{R}^p$. In our work, we obtain the LBS by solving a BOLP relaxation and maintain a global UBS by storing the best solutions known so far. To evaluate the subproblems, the three basic fathoming rules are infeasibility, integrity and dominance. Particularly in the MO context, additional to the classical variable branching, it is possible to also branch on subproblems by bounding on objectives, which is known as the Pareto branching [8, 9].

3 Bi-objective Branch&Cut algorithm

To tighten the LBS in purpose of helping nodes fathoming, it is natural to generate and apply valid inequalities that cuts off the current fractional points belonging to the LBS, and provides better ones. A very first naive approach is to find a valid inequality for each fractional extreme point in a LBS, which nevertheless may cause a very heavy LP formulation due to the exponentially large non-dominated set. Consequently, we prefer to produce stronger valid inequality cutting off a set of fractional points simultaneously. To address this *multi-point separation problem*, it is required to separate the convex combination of a set of fractional points from the convex hull of feasible region. In bi-objective case, the BOLP bound set is convex and all points are naturally ordered in criteria space, thus this problem reduces to search a valid cut violated only by the left-most and the right-most points in the given point set. Subsequently we design a multi-point cutting plane who iterates on the LBS, calls the multi-point separator and re-optimize the BOLP problem with found valid cuts.

Our first preliminary experiments show the strong efficiency of the multi-point separator in BOB&C algorithm on bi-objective binary linear problems.

Interestingly the set of p objective functions can be seen as a linear projection from the decision space to the criterion space, we propose first polyhedral results about the relation between valid cuts in criteria space and cuts in decision space.

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