What is the optimal cutoff grade for multiple minerals?

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1 Introduction

The standard practice in mine planning is to first define the contour of a mine by solving the ultimate pit problem, and then to perform a cutoff grade optimization to define at each moment in time which material should be mined and processed; the rest is considered waste. The seminal work of K. Lane \cite{1} established a unified framework to perform cutoff grade optimization, taking into account economic factors, production capacities, and the time value of money. The algorithm proposed in \cite{1} is widely used in commercial software for the mining industry, and its optimality has been characterized by \cite{2}.

We study deposits with multiple minerals with special attention to the two-dimensional case. Mines that contain more than one economic mineral include copper-gold, lead-zinc, copper-lead-zinc, among others. Our main contribution is to show that, in two dimensions, the solution that maximizes the operations’ net present value is formed by grade-pairs that can be defined as the region above a line. We also show that in \(n\) dimensions the optimal cutoff surface is a hyperplane.

2 The mining scheduling problem

We consider the mining operation as a succession of three stages: mining, concentrating, and refining. We say that a function \(\lambda : [0, \bar{g}_1] \times [0, \bar{g}_2] \rightarrow [0, 1]\) is a grade density function describing an homogeneous orebody if it is Lebesgue-integrable with \(\int_0^{\bar{g}_1} \int_0^{\bar{g}_2} \lambda(g_1, g_2) dg_1 dg_2 = 1\), where \(\bar{g}_i\) is the highest grade of mineral \(i\) present in the mine. We define set of admissible grade-pairs, denoted by \(\Omega \subseteq [0, \bar{g}_1] \times [0, \bar{g}_2]\), as the set of grade-pairs \((g_1, g_2) \in \Omega\) that is sent to the concentrator; the rest is waste. Figure 1 shows some of the level sets of the grade density function \(\lambda\) together with the set \(\Omega\) in gray.

FIG. 1 – A cutoff line, with admissible grade-pairs in gray.
We define $Q_{m,t}$ as the amount of material to be extracted from the deposit at time $t$, $Q_{c,t}$ is the amount of extracted material to be sent to the concentrator, and $Q^{1}_{r,t}$ and $Q^{2}_{r,t}$ are the amount of minerals 1 and 2 to be refined, respectively. They can be characterized as

$$Q_{c,t} = Q_{m,t} \int \int_{\Omega} \lambda(g_1, g_2) dg_1 dg_2, \quad Q^{i}_{r,t} = Q_{m,t} z_{i} \int \int_{\Omega} g_{i} \lambda(g_1, g_2) dg_1 dg_2, \quad i = 1, 2,$$

where $z_{i}$ is the recovery rate. We can write the mining schedule problem in two dimensions as

$$V(U_{o}) = \max_{\Omega, Q_{m}, Q_{c}, Q_{r}} \left\{ \sum_{t=0}^{T} \delta^{t} [b_{1} Q^{1}_{r,t} + b_{2} Q^{2}_{r,t} - c Q_{c,t} - m Q_{m,t} - f w_{t}] \right\} \text{ s.t. } Q_{c,t} \leq w_{t} C, Q^{i}_{r,t} \leq w_{t} R_{i}, \quad \text{for } i = 1, 2, Q_{m,t} \leq w_{t} M, w_{t} \in [0, 1], \sum_{t=1}^{T} Q_{m,t} \leq U_{o},$$

where $(M, C, R_{i})$ and $(m, c, r_{i})$ are upper limits and unit costs, respectively, in the mining operation, $s_{i}$ is the sale price, $f$ is a fixed cost, $d$ is the discount rate, $\delta = 1/(1 + d)$, $T$ is the time horizon, $w_{t}$ is the percentage of the time period over which the mine is operational, and $U_{0}$ is the material left to be extracted.

### 2.1 Bilevel reformulation of problem (1)

$$V(U) = \max_{Q_{m} \in [0, \min\{wM, U\}], w \in [0, 1]} \{v(Q_{m}, w) + \delta V(U - Q_{m})\},$$

where

$$v(Q_{m}, w) = Q_{m} \max_{\Omega} \left\{ \int \int_{\Omega} (b_{1} g_{1} + b_{2} g_{2} - c) \lambda(g_1, g_2) dg_1 dg_2 \right\} - m Q_{m} - f w \quad \text{s.t.} \int \int_{\Omega} \lambda(g_1, g_2) dg_1 dg_2 \leq w C / Q_{m}, \quad z_{i} \int \int_{\Omega} g_{i} \lambda(g_1, g_2) dg_1 dg_2 \leq w R_{i} / Q_{m}, \quad i = 1, 2.$$

### 2.2 Theorem

The optimal curve $\Omega$ is a line in 2-D, and a hyperplane in $n$-dimensions.

**Idea of the proof:** We use Green’s Theorem to convert integrals of areas into line integrals, and then Euler-Lagrange equations to derive the necessary conditions the curve has to satisfy. Considering all possible active constraint cases, we concluded the optimal curve is a line. In [3] we extended the results for the $n$-dimensional case using the generalized Stokes Theorem and showed that the optimal solution is a hyperplane.

### Références

