Computing Approximated Nash Equilibria for Integer Programming Games with Nonlinear Payoffs

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1 Problem Description

The Nash equilibrium (NE) is a widely accepted solution concept for non-cooperative games. An NE models the following notion of stability: a profile of strategies is an NE if no players has incentive to unilaterally deviate from it. When players are allowed to deviate from an NE up to a small quantity $\delta \geq 0$, we obtain a relaxed solution concept of a game: $\bar{x}$ is a δ-equilibrium of a game G if for each player p:

$$\Pi^p(\bar{x}) + \delta \geq \max_{x^p \in X^p} \Pi^p(x^p, \bar{x}^{-p})$$

where x^{-p} denotes the variables x of the players other than p, Π^p and X^p are the payoff function and the set of feasible strategies for player p, respectively. In other words, $\bar{x}$ is a best response up to δ for each player p. If $\delta = 0$, we recover the definition of NE.

The goal of our work is to establish a method for finding effectively δ-equilibria in a subclass of integer programming games (IPG), namely, simultaneous, complete-information and non-cooperative games among N players where each player p solves the optimization problem:

$$(\mathcal{P}_p)\begin{cases} \max_{x^p} & \Pi^p(x^p, x^{-p}) = f^p(x^p) + g^p(x^p, x^{-p}) \\ s.t. & A^p x^p \leq b^p \\ & x^p \in \mathbb{R}_+^{n_p} \times \mathbb{N}^{k_p} \end{cases}$$

where f^p and g^p are continuous functions (not necessarily linear), and A^p and b^p are a matrix and a vector of appropriate dimension, respectively. The supply chain network game described in [5] is an example of this subclass, with a non-linear function f^p.

2 State-of-the-art

As IPGs are not finite games in general, Nash's theorem can not be applied to prove the existence of an NE. It was shown in [1] that an IPG G with the feasible set of each player nonempty and bounded has an NE, in particular, a mixed Nash equilibrium. Thus, G has an equilibrium in nontrivial cases. A recent algorithm called SGM [2] can find δ-NE to IPGs for any $\delta > 0$, with the restraint that an exact method to compute the best responses $(\mathcal{P}_p)$ is available. To solve a game, the algorithm samples strategies from X^p for each player to create a finite game, it computes an NE of this finite game, and it checks if there exists a strategy in the original continuous game that can improve one of the player's payoff. If there exists such

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a strategy, it is added to the finite game which is solved again. Those operations are repeated until no improvement to the players' payoff is possible.

One of the drawback of this method is that its computation time heavily depends on the time needed to compute a best response. In particular, whenever Π^p is a non-convex function, the computation of a best response corresponds to a mixed integer nonlinear problem (MINLP), which may be computationally demanding. A common solution to this too high demand is to trade speed with precision : building an MILP approximating the MINLP by replacing the nonlinear functions with piecewise linear functions (PWL) [4]. Moreover, recent works focused on a crucial point of such an approximation, which is to control its precision [7; 6; 3]. It consists in approximating a function f on domain D with a PWL $\hat{f}$ such that $|\hat{f}(x) - f(x)| < \delta$ for all $x \in D$. In this case, $\hat{f}$ is called a δ-absolute approximation of f.

3 Nash Equilibrium Approximation Methodology

Our work exploits a link between absolute approximation of functions, Nash equilibria and δ-equilibria described in Proposition 1.

Proposition 1. *Let G be an integer programming game among N players where the optimization problem of each player p is of the form $\mathcal{P}_p$. Suppose now that $\hat{G}$ is also a game with N players, and the optimization problem of each player $p = 1, ..., N$ is the same as that of player p in G except for the payoff $\hat{f}(x^p) + g^p(x^p, x^{-p})$ where $\hat{f}$ is a δ-absolute approximation of f in X^p with X^p the feasible set of player p. Then, we have that a Nash equilibrium $\hat{\sigma}$ of $\hat{G}$ is a 2δ-equilibrium of G.*

This proposition basically says that approximating a game G into $\hat{G}$ by replacing the nonlinear functions f with δ-absolute approximations $\hat{f}$ of f and then finding an NE $\hat{\sigma}$ of $\hat{G}$ implies that $\hat{\sigma}$ is also a 2δ-equilibrium of G. In light of the previous result, we propose an algorithm to compute δ-equilibria to an IPG G with nonlinear functions f^p and functions $g^p(x^p, x^{-p})$ linear or quadratic in x^p using the SGM algorithm: approximate f^p with an absolute approximation error of $\frac{1}{2}\delta$ with PWL $\hat{f}^p$, and use the SGM algorithm to find a Nash equilibrium $\hat{\sigma}$ of the resulting game $\hat{G}$, which is also a δ-equilibrium of G according to Proposition 1.

Theoretical results and preliminary computational results will be presented.

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