The d-interaction index in MCDA

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Abstract. In the context of preference modeling, in MCDA, we introduce a new interaction index based on a distance, in order to better model the interactions among criteria.

Keywords: MCDA, independence, interaction, Choquet integral, Kendall distance

1 Introduction

Multiple Criteria Decision Analysis (MCDA) aims at representing numerically the preferences of a Decision Maker (DM), by using an appropriate aggregation function. The additive model is usually used when the DM preferences satisfy the independence axiom. If not, the Choquet integral [1, 3] model can be used, allowing to capture interactions among criteria. However, given a preference information of the DM, the sign of the interaction, of a subset of criteria, is not always stable [2]. This leads to some misinterpretations of this notion. We introduce a new interaction index, named the d-interaction, directly linked to the preferences and the independence axiom, which is null if and only if there is no interaction among criteria.

2 Settings

Let $X = X_1 \times X_2 \times \ldots \times X_n$ be a set of alternatives evaluated on a finite set of $n \geq 3$ criteria, $N = \{1, 2, \ldots, n\}$ ($n \geq 3$), where X_i refers to the discrete set of attribute i, $i = 1, \ldots, n$. A marginal utility function $u_i : X_i \rightarrow \mathbb{R}_+$, is associated to each attribute X_i. Given any $T \subset N$, we set $X_T = \prod_{i \in T} X_i$ and $z = (x_S, y_{N\setminus S})$ means that $z_i = x_i$ if $i \in S$ and $z_i = y_i$ otherwise.

Definition 1 (Independence Axiom)

1. *S is preference independent of $N\setminus S$, w.r.t. $\succsim_X$, if for all $x_S, x'_S \in X_S$, $a_{N\setminus S}, b_{N\setminus S} \in X_{N\setminus S}$,*

$$(x_S, a_{N\setminus S}) \succsim_X (x'_S, a_{N\setminus S}) \Rightarrow (x_S, b_{N\setminus S}) \succsim_X (x'_S, b_{N\setminus S}) \tag{1}$$

2. *The preference relation $\succsim_X$ is said to satisfy the independence axiom if for every subset $S \subseteq N$, S is preference independent of $N \setminus S$.*

Definition 2 *Given an alternative $x := (x_1, ..., x_n)$ of X, the expression of the Choquet integral of x, w.r.t. a 2-additive capacity, is given by*

$$C_\mu(u_1(x_1), \ldots, u_n(x_n)) = \sum_{i=1}^{n} \phi_i^\mu \ u_i(x_i) - \frac{1}{2} \sum_{\{i,j\} \subseteq N} I_{ij}^\mu \ |u_i(x_i) - u_j(x_j)|$$

where $I_{ij}^\mu = \mu(\{i, j\}) - \mu(\{i\}) - (\{j\})$ is the interaction index between i and j and ϕ_i^μ is the Shapley value of i.

Example 1 *Let us assume that the scale $[0, 20]$ of the evaluation of four students, corresponds to the utility function associated to each subject, i.e.,*

1: Mathematics (M)	*2: Statistics (S)*	*3: Language (L)*
$u_1(16) = 16$	$u_2(13) = 13$	$u_3(7) = 7$
$u_1(16) = 16$	$u_2(11) = 11$	$u_3(9) = 16$
$u_1(6) = 6$	$u_2(13) = 13$	$u_3(7) = 7$
$u_1(6) = 6$	$u_2(11) = 11$	$u_3(9) = 16$

It is not difficult to see that the preference information $a \succ_X b \succ_X c \succ_X d$ is representable by an additive model, but also by a 2-additive Choquet integral model in which all the interactions indices are not null (see Table 1).

$\mu(\{1\})$	0.1
$\mu(\{2\})$	0.5
$\mu(\{3\})$	0.5
$\mu(\{1,2\})$	0.9
$\mu(\{1,3\})$	0.7
$\mu(\{2,3\})$	0.5

I^μ_{12}	0.3
I^μ_{13}	0.1
I^μ_{23}	-0.5

ϕ^μ_1	0.3
ϕ^μ_2	0.4
ϕ^μ_3	0.3

$C_\mu(a)$	12.7
$C_\mu(b)$	11.3
$C_\mu(c)$	9.5
$C_\mu(d)$	8.5

Table 1. $a \succ_X b \succ_X c \succ_X d$ are representable by a 2-additive Choquet integral C_μ.

A misinterpretation of the interaction index might led to believe that in a Choquet model, independence is equivalent to having null interactions. However, as shown in the previous example, preferences may verify the independence axiom but been representable by a Choquet model with interactions different from zero. This finding motivates the introduction of a new concept of interaction for which nullity coincides with independence.

3 A new interaction index

Definition 3 *Let d be a distance defined on the set of all binary relations and S be a subset of N. The d-interaction index between criteria is defined by :*

$$I^d(S) = \sum_{\{y_{N\setminus S}, z_{N\setminus S}\} \subseteq X_{N\setminus S}} d(\succsim_S^{y_{N\setminus S}}, \succsim_S^{z_{N\setminus S}})$$

The d-interaction index of a coalition S of criteria captures through the distance the effect of $N \setminus S$ over S.

Theorem 1. *A preference relation $\succsim_X$ satisfies the independence axiom if and only if $I^d(S) = 0$ for all $S \in 2^N$.*

Theorem 2. *Let $\succcurlyeq$ be a preference relation and d a distance over binary relations. A necessary and sufficient condition under which $I^d(S) = 0$ for all $S \in 2^N$, is that, for all $i \in N$, $I^d(N \setminus i) = 0$*

Given two binary relations R_1 and R_2, the Kendall distance between R_1 and R_2, denoted by $D_K(R_1, R_2)$ is defined by : $D_K(R_1, R_2) = |R_1 \setminus R_2| + |R_2 \setminus R_1|$. In the above example, it is not difficult to see that the preferences $b \succ_X a$ and $c \succ_X d$ are not representable by an additive model. Therefore, we have $I^d(23) > 0$ since $\succsim_{23}^{a_1} \neq \succsim_{23}^{a_2}$, i.e., these preferences do not satisfy the independence axiom.

References

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