Small quasi-kernels in split graphs

Hélène Langlois[1], Frédéric Meunier[1], Romeo Rizzi[2], Stéphane Vialette[3]

[1] CERMICS, École des Ponts ParisTech, 77455 Marne-la-Vallée, France
{helene.langlois,frederic.meunier}@enpc.fr

[2] Department of Computer Science, Università di Verona, 37129 Verona, Italy
romeo.rizzi@univr.it

[3] LIGM, Univ Gustave Eiffel, 77454 Marne-la-Vallée, France
stephane.vialette@univ-eiffel.fr

Keywords : *Quasi-kernel, split graphs, digraph, computational complexity.*

1 Introduction

Let D be a digraph. A *kernel* K is a subset of vertices that is independent (*i.e.*, all pairs of distinct vertices of K are non-adjacent) and such that, for every vertex $v \notin K$, there exists $w \in K$ with $(v, w) \in A$. Kernels were introduced in 1947 by von Neumann and Morgenstern [9]. It is now a central notion in graph theory and has important applications in relations with colorings, perfect graphs, game theory and economics, logic, etc. Chvátal proved that deciding whether a digraph has a kernel is NP-complete [3] and the problem is equally hard for planar digraphs with bounded degree [5].

Chvátal and Lovász [2] introduced quasi-kernels in 1974. A *quasi-kernel* in a digraph is a subset Q of vertices that is independent and such that every vertex of the digraph can reach some vertex in Q via a directed path of length at most two. In particular, any kernel is a quasi-kernel. Yet, unlike what happens for kernels, every digraph has a quasi-kernel. Chvátal and Lovász provided a proof of this fact, which can be turned into an simple polynomial-time algorithm.

In 1976, Erdős and Székely [4] conjectured that every sink-free digraph D has a quasi-kernel of size at most $|V(D)|/2$. This question is known as the *small quasi-kernel conjecture*. So far, this conjecture is only confirmed for narrow classes of digraphs. In 2008, Heard and Huang [7] showed that every digraph D has two disjoint quasi-kernels if D is a sink-free tournament or a transitive digraph. In particular those graphs respect the small quasi-kernel conjecture. Recently, Kostochka et al. [8] renewed the interest in the small quasi-kernel conjecture and proved that the conjecture holds for orientations of 4-colorable graphs (in particular, for planar graphs).

A split graph is a graph in which the vertices can be partitioned into a clique and an independent set. This class seems to play an important role in the study of small quasi-kernels since the only examples of oriented graphs having no two disjoint quasi-kernels contain the orientation of a split graph constructed by Gutin et al. [6].

2 Computational hardness

We first show that one cannot confine the seemingly inevitable combinatorial explosion of computational difficulty to the size of the sought quasi-kernel. We consider the problem Quasi-Kernel of deciding wether a graph admits a quasi-kernel smaller than a given size.

Proposition 2.1. Quasi-Kernel *is* W[2]*-complete when the parameter is the size of the sought quasi-kernel even for orientations of split graphs.*

Proposition 2.2. Quasi-Kernel *for orientations of split graphs is FPT for parameter* $|T|$ *or parameter* $k + |I|$*, where* T *is the set of vertices in the clique-part,* I *is the set of vertices in the independent-part and* k *is the size of the sought quasi-kernel.*

3 Not too big quasi-kernels in split graphs

A one-way split digraph consists of an independent set X and a semicomplete digraph on Y and arcs from X to Y. The small quasi-kernel conjecture has been proved in the particular case of one-way split graphs [1], but remains unproved for general split graphs. We provide a weaker bound in the latter case.

Theorem 3.1. *Every sink-free split digraph* D *admits a quasi-kernel of size at most* $\frac{3}{4}|V(D)|$*.*

A split graph is *complete* if every vertex in the independent-part is adjacent to every vertex in the clique-part. A consequence of the next theorem is that Quasi-Kernel is polynomial-time solvable for complete split digraphs.

Theorem 3.2. *Let* D *be an orientation of a complete split graph. If* D *has a sink, then there is a unique minimum-size quasi-kernel, which is formed by all sinks. If* D *has no sink, then the minimum size of a quasi-kernel is at most two.*

Orientations of complete split graphs always have two disjoint quasi-kernels when there is no sink. This is a consequence of a result by Heard and Huang [7]. The existence of two disjoint quasi-kernels for this class of digraphs can thus trivially be decided in polynomial time. Their proof provides actually a polynomial-time algorithm for finding such quasi-kernels.

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