Optimization problems in graphs with locational uncertainty

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1 Introduction

Research in combinatorial optimization has provided efficient algorithms to solve many complex discrete decision problems, providing exact or near-optimal solutions in reasonable amounts of time. The applications are countless, ranging from logistics (network design, facility location, ...) to scheduling. In this paper, we are interested in the class $\mathcal{S}$ of deterministic combinatorial optimization problems that amount to selecting a feasible set of edges in a given graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ and that minimizes the sum of edge-weights. Any $\Pi \in \mathcal{S}$ represents a specific problem, such as the shortest path or the minimum spanning tree problem. We consider further that $\mathbb{G}$ is a spatial graph embedded into a given metric space $(\mathcal{M}, d)$. Each vertex i is assigned a position $u_i \in \mathcal{M}$ so the weight of each edge $\{i, j\}$ is given by its distance $d(u_i, u_j)$. Denoting by $\mathcal{X} \subseteq \{0, 1\}^{|\mathbb{E}|}$ the set of feasible vectors for a given instance, any $\Pi \in \mathcal{S}$ corresponds to a combinatorial optimization problem of the form

$$\min_{x \in \mathcal{X}} \sum_{\{i,j\} \in \mathbb{E}} x_{ij} d(u_i, u_j). \tag{Π}$$

Problem Π encompasses many applications, such as network design and facility location. These are typically subject to data uncertainty, be it because of the duration of the decision process, measurement errors, or simply lack of information. One successful framework that has emerged to address uncertainty is robust optimization [2], where the uncertain parameters are modeled with convex sets such as polytopes, or with finite sets of points. Many authors have focused more particularly on robust discrete optimization problems, see [4, 5] and the references therein.

We enter this framework by considering the model where the positions of the vertices are subject to uncertainty, therefore impacting the distances among the vertices. The resulting problem thus seeks to find a feasible set of edges that minimizes its worst-case sum of distances. Formally, we introduce for each vertex $i \in \mathbb{V}$ the set of possible locations as the uncertainty set $\mathcal{U}_i \subseteq \mathcal{M}$ of cardinality $\sigma_i = |\mathcal{U}_i|$. We consider that there is no correlation between the positions of the different vertices, so a scenario is given by the tuple $u = (u_1, \ldots, u_{|\mathbb{V}|})$ which belongs to the set $\mathcal{U} = \times_{i \in \mathbb{V}} \mathcal{U}_i$. Then, given $\Pi \in \mathcal{S}$, we study in this paper the *locational robust counterpart* of problem Π, formally defined as

$$\min_{x \in \mathcal{X}} \max_{u \in \mathcal{U}} \sum_{\{i,j\} \in \mathbb{E}} x_{ij} d(u_i, u_j). \tag{LocRob-Π}$$

We also devote a particular attention to evaluating the objective function of LocRob-Π, often called the adversarial problem

$$\max_{u \in \mathcal{U}} \sum_{\{i,j\} \in \mathbb{E}} x_{ij} d(u_i, u_j). \tag{adversarial}$$

We underline that we focus throughout on finite uncertainty sets. However, our setting encompasses polyhedral uncertainty sets whenever the distance function is convex.

2 Contributions

Let us denote by $G(x) = (V(x), E(x))$ the subgraph induced by x, where

$$E(x) = \{\{i, j\} \in \mathbb{E} \mid x_{ij} = 1\}$$

and

$$V(x) = \{i \in \mathbb{V} \mid \exists e \in E(x) : i \in e\}.$$

In this context, we can summarize our contributions as follows (see [3] for details) :

- We prove that LocRob-Π is $\mathcal{NP}$-hard even when $\mathcal{X}$ consists of all $s-t$ paths and $(\mathcal{M}, d)$ is the one-dimensional Euclidean metric space or when $\mathcal{X}$ consists of all spanning trees of $\mathbb{G}$. These results illustrate how the nature of LocRob-Π fundamentally differs from the classical min-max robust problem with cost uncertainty, which is known to be polynomially solvable whenever the costs lie in independent uncertainty sets [1].
- We provide a general cutting-plane algorithm for LocRob-Π. We further show that problem adversarial is $\mathcal{NP}$-hard and provide two algorithms for computing adversarial. One is based on integer programming formulations while the other one relies on a dynamic programming algorithm that involves the threewidth of $G(x)$.
- We leverage the above dynamic programming to provide a compact formulation for the problem when any $G(x)$ contains only stars (or unions of stars). We can, in theory, extend that idea to trees, albeit presenting poor numerical performance.
- We propose a conservative approximation of the problem that uncouples $\mathcal{U}$ into its projections $\mathcal{U}_i$, $i \in \mathbb{V}$. In the case of Euclidean metric spaces, this approximation leads to mixed-integer second-order conic reformulations, and turns out to be equivalent to the affine decision rule reformulation proposed by [6].
- We compare the exact cutting plane algorithm numerically with the above conservative approximation and simple deterministic reformulations. The benchmark is composed of two families of instances. The first family includes Steiner tree instances that illustrate subway network design. The second one is composed of strategic facility location instances.

Références

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