Envy-free division of multi-layered cakes

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1 Introduction

Consider n agents having preferences over the connected pieces of a cake, identified with the interval $[0, 1]$. A classical theorem ensures under mild conditions that it is possible to divide the cake into n connected pieces and assign these pieces to the agents in an *envy-free* manner, i.e, no agent strictly prefers a piece that has not been assigned to her. Motivated by the assignments of time slots to a given set of activities, Hosseini, Igarashi, and Searns [4] have introduced the multi-layered cake-cutting problem. There, n agents divide a cake formed by m different layers (m copies of $[0, 1]$) so that each piece of any layer is connected and the pieces of different layers assigned to a same agent, which form a *layered piece*, are *non-overlapping*, i.e., have disjoint interiors. For the special case of two layers and two agents, Hosseini et al. showed the existence of such an envy-free division. For other values of m and n with $m \leqslant n$ (otherwise, there is no solution), it has remained an open question whether there exists such an envy-free multi-division, even in the special case when $m = 2$ and $n = 3$. In this work, we address the problem of existence of envy-free division of a multi-layer cake, in a slightly more general setting than that of Hosseini et al.: with a birthday agent and with groups. The version with a birthday agent [1] looks for a division such that whichever piece the *birthday agent* selects, there is an envy-free assignment of the remaining pieces to the other agents. The group version [8] takes a number q of groups in input and looks for a partition of the n agents into q groups and an envy-free division into q layered pieces between the q groups.

2 Existence result

We prove that envy-free divisions exist for multi-layer cakes provided that n is a prime power, thereby settling the open question Hosseini et al. for many cases (included the special case $m = 2$ and $n = 3$). Actually, we prove a more general result (Theorem 1) including a birthday agent and groups. The assumption "with closed preferences" is a technical yet natural assumption that can be seen as a continuity assumption and whose precise definition is omitted in this abstract.

Theorem 1 *Consider an instance of the multi-layered cake-cutting problem with m layers and n agents, $m \leqslant n$, with closed preferences. Let q be an integer non-smaller than m. If q is a prime power, then there exists a connected and non-overlapping multi-division into q layered pieces so that no matter which layered piece the birthday agent chooses, the remaining agents can be assigned to the layered pieces while satisfying the following two properties:*

- *each of the remaining agents is assigned to one of her preferred layered pieces.*
- *the number of agents assigned to each layered piece, including the birthday agent, differs by at most one.*

The proof relies on a general Borsuk–Ulam-type theorem, originally proved by Volovikov, and recently applied by Jović, Panina, and Živaljević [5] on the envy-free division problem of a (single-layer) cake. Since the agents might prefer zero-length pieces, the special case $m = 1$ and $q = n$ of Theorem 1 boils down to recent results for the single-layer cake with non-necessarily hungry agents [2, 6, 7], which do not hold if q is not a prime power.

3 Algorithms

To state and prove results about algorithms, we specialize the preferences to "valuation functions," whose precise description is omitted. We design a fully polynomial-time approximation scheme (FPTAS) for the two-layer three-group case.

Theorem 2 *Consider an instance of the two-layered cake-cutting problem with n agents, $n \geqslant 2$, whose valuation functions satisfy monotonicity and the Lipschitz condition with constant K. Then, for any $\varepsilon > 0$, one can find in time $O(n \log^2 \frac{K}{\varepsilon})$ a connected and non-overlapping multi-division into three layered pieces where no matter which layered piece the birthday agent chooses, the remaining agents can be assigned to the layered pieces while satisfying the following two properties:*

- *each of the remaining agents is assigned to one of her ε-approximate preferred layered pieces.*
- *the number of agents assigned to each layered piece differs by at most one.*

When there is only one layer, the algorithm can handle the case where the sizes of the groups can be set arbitrarily, which corresponds to an FPTAS version of the three-group case of a recent result by Segal-Halevi and Suksompong [8]. The special case $n = 3$, with no birthday-agent, of this FPTAS is a classical result by Deng et al. [3]. But this FPTAS follows neither from the work by Segal-Halevi and Suksompong nor from the work by Deng et al.

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